Quarter Power Law Scaling

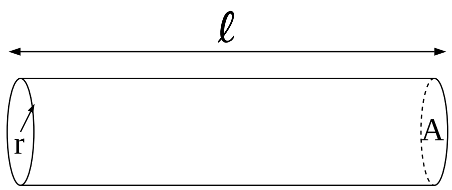
Capillaries are very small, cylindrical blood vessels through which blood cells flow single-file. Since blood cells are essentially the same size for all animals, it follows that capillaries are essentially the same size for all animals.

ℓ_C length of capillary

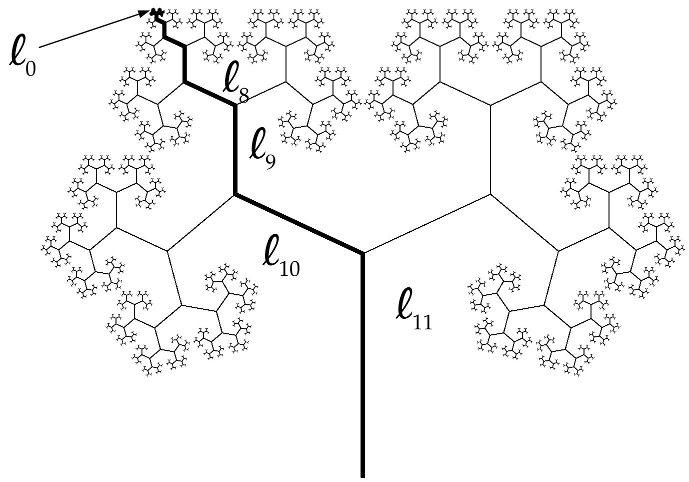
r_C radius of capillary

A_C cross-sectional area of capillary $= \pi r_C^2$

V_C volume of capillary $= \pi r_C^2 \ell_C = A_C \ell_C$



The arterial tree is *fractal*



N number of branching levels (equals 11 in above diagram)

b the branching number: the number of off-shoots at a junction (equals 2 in the above diagram)

D the fractal dimension of the arterial tree

The total number of capillaries in the arterial network is

$$N_C = b^N$$

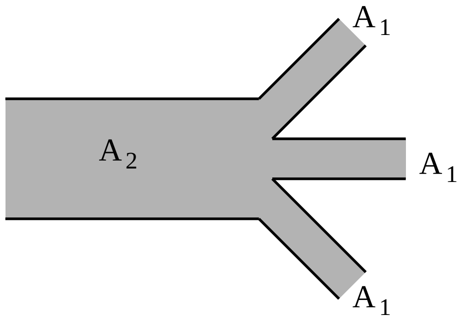
Because the network is fractal, the lengths of the branches are related.

$$\begin{aligned} \ell_1 &= b^{\frac{1}{D}} \ell_C \\ \ell_2 &= b^{\frac{1}{D}} \ell_1 \\ &= b^{\frac{2}{D}} \ell_C \\ \ell_3 &= b^{\frac{3}{D}} \ell_C \\ \ell_N &= b^{\frac{N}{D}} \ell_C \end{aligned}$$

The arterial tree uniformly fills space within the organism so that all cells are close to a capillary. This means that the fractal dimension must be equal to the dimension of space:

$$D = 3$$

The arterial tree is constructed to minimize energy loss as blood is pumped through it. If the arterial tree is assumed to be rigid, and blood flow is assumed to be steady, then cross-sectional area is preserved at each branch point.



$$
\begin{aligned}
A_1 &= bA_C \\
A_2 &= bA_1 \\
&= b^2 A_C \\
A_3 &= b^3 A_C \\
A_N &= b^N A_C
\end{aligned}
$$

The volume of each branch of the tree at some level n is equal to

$$\begin{aligned} V_n &= A_n \ell_n \\ &= \left(b^n A_C\right)\left(b^{\frac{n}{3}} \ell_C\right) \\ &= b^{\frac{4}{3}n} A_C \ell_C \\ &= b^{\frac{4}{3}n} V_C \end{aligned}$$

The total volume of blood, V_B, in the organism is equal to the blood in all the branches of the arterial tree

$$\begin{aligned} V_B &= V_N + bV_{N-1} + b^2 V_{N-2} + \ldots + b^{N-2} V_2 + b^{N-1} V_1 + b^N V_C \\ &\sim V_N \\ &\sim b^{\frac{4}{3}N} V_C \\ &\sim \left(b^N\right)^{\frac{4}{3}} V_C \\ &\sim N_C^{\frac{4}{3}} V_C \end{aligned}$$

The volume of blood is directly proportional to the mass of the organism, M. So, the mass of of the organism is related to the number of capillaries and the size of the capillaries by the equation

$$M \sim N_C^{\frac{4}{3}} V_C$$

which can be re-written to solve for N_C in terms of M and V_C:

$$N_C \sim \left(\frac{M}{V_C}\right)^{\frac{3}{4}}$$